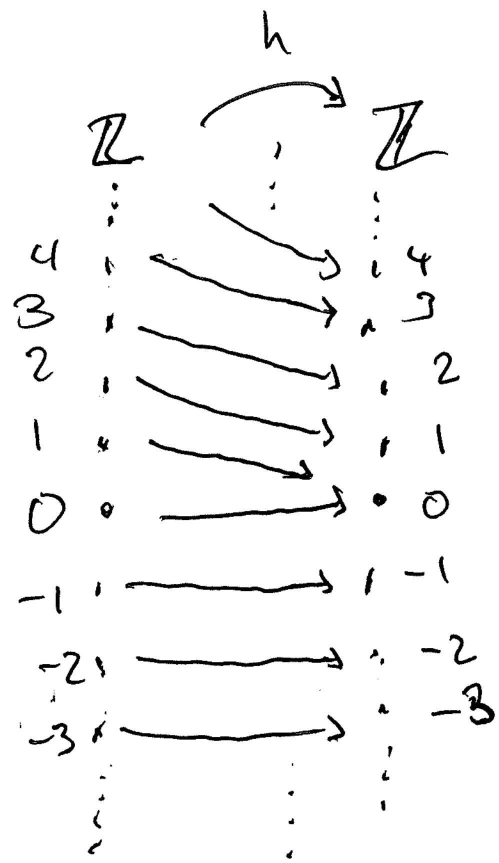
PROJECTED WRITTEN NOTES FROM THE M 325K LECTURE ON TUESDAY, APRIL 16, 2024, ON SECTION 7.3, on COMPOSITION OF FUNCTIONS, INVERSE FUNCTIONS and the TEST FOR A ONE-TO-ONE CORRESPONDENCE - Thm (NIB) 10. CLASS #25

Problem: Define a function $h : \mathbb{Z} \to \mathbb{Z}$ such that h is onto, but h is not one-to-one.

Sol'n:



h is onto, but h is not one-to-one because $h(0) = h(1) = 0$ and $0 \neq 1$.

Define function $h : \mathbb{Z} \to \mathbb{Z}$ as follows:

For every $n \in \mathbb{Z}$,

$$h(n) = \begin{cases} n & \text{if } n \leq 0 \\ n-1 & \text{if } n > 0 \end{cases}$$

<u>**One-to-One Functions & Onto Functions**</u>

Official In-the-book Definitions: Let F be a function from a set X to a set Y.

F is *one-to-one* (or *injective*) $\Leftrightarrow$ For every u and v in X,

If F(u) = F(v), Then u = v

Also,

F is *one-to-one* (or *injective*) $\Leftrightarrow$ For every u and v in X,

If $u \neq v$, Then $F(u) \neq F(v)$.

F is *onto* (or *surjective*) $\Leftrightarrow$ For every element $y \in Y$,

there exists some $x \in X$ such that $F(x) = y$.

F is *a one-to-one correspondence* (or a *bijection*) from X to Y

$\Leftrightarrow$ $F: X \to Y$ is both a one-to-one function and an onto function.

Memorize the above definitions for their use in writing proofs, but a more intuitive definition of these terms is useful and is as follows:

Let $f : X \to Y$ be a function.

Function f is ... $\left\{ \begin{array}{c} \underline{\text{onto}} \\ \\ \underline{\text{one-to-one}} \\ \\ \underline{\text{one-to-one and onto}} \end{array} \right\}$ if each element of Y is the image of ... $\left\{ \begin{array}{c} \text{at least one} \\ \\ \text{at most one} \\ \\ \text{exactly one} \end{array} \right\}$ element of X

If $f : X \to Y$ is one-to-one and onto, then the *inverse function* $f^{-1} : Y \to X$ exists and $f^{-1}(y) = x$ if and only if $f(x) = y$, for all x in X and all y in Y.

and so by the one-to-oneness of the exponential function (property 7.2.5),

$$uv = w.$$

Substituting from (1), (2), and (3) gives that

$$(\log_b c)(\log_c x) = \log_b x.$$

And dividing both sides by $\log_b c$ (which is nonzero because $c \neq 1$) results in

$$\log_c x = \frac{\log_b x}{\log_b c}.$$ ■

Example 7.2.6 Computing Logarithms with Base 2 on a Calculator

In computer science it is often necessary to compute logarithms with base 2. Most calculators do not have keys to compute logarithms with base 2 but do have keys to compute logarithms with base 10 (called **common logarithms** and often denoted simply log) and logarithms with base e (called **natural logarithms** and usually denoted ln). Suppose your calculator shows that $\ln 5 \cong 1.609437912$ and $\ln 2 \cong 0.6931471806$. Use Theorem 7.2.1(d) to find an approximate value for $\log_2 5$.

Solution By Theorem 7.2.1(d),

$$\log_2 5 = \frac{\ln 5}{\ln 2} \cong \frac{1.609437912}{0.6931471806} \cong 2.321928095.$$ ■

One-to-One Correspondences

Consider a function $F\colon X \to Y$ that is both one-to-one and onto. Given any element x in X, there is a unique corresponding element $y = F(x)$ in Y (since F is a function). Also given any element y in Y, there is an element x in X such that $F(x) = y$ (since F is onto) and there is only one such x (since F is one-to-one). Thus, a function that is one-to-one and onto sets up a pairing between the elements of X and the elements of Y that matches each element of X with exactly one element of Y and each element of Y with exactly one element of X. Such a pairing is called a *one-to-one correspondence* or *bijection* and is illustrated by the arrow diagram in Figure 7.2.5. One-to-one correspondences are often used as aids to counting. The pairing of Figure 7.2.5, for example, shows that there are five elements in the set X.

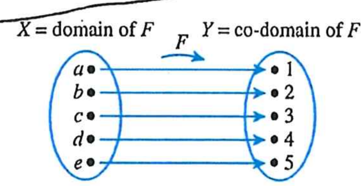


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Figure 7.2.5 An Arrow Diagram for a One-to-One Correspondence

• Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F\colon X \to Y$ that is both one-to-one and onto.

<u>Definition:</u> Let $f: X \to Y$ be a function such that

f is a one-to-one correspondence

(i.e. f is one-to-one and onto).

Define function $f^{-1}: Y \to X$ as follows:

For all $y \in Y$, $f^{-1}(y) = x$ if and only if $f(x) = y$.

That is, $f^{-1}(y_0)$ = That unique $x_0 \in X$ such that $f(x_0) = y_0$.

FACT: A function f has an inverse function f^{-1}

$\Longleftrightarrow$

f is a one-to-one correspondence.

<u>Ex:</u> (FINDING the Inverse function)

Let function $g: \mathbb{R} \to \mathbb{R}$ be defined as follows:

for all $x \in \mathbb{R}$, $g(x) = \sqrt[3]{5x-1}$

<u>FACT:</u> Function g is a one-to-one correspondence.

So, g^{-1} exists as a function: $g^{-1}: \mathbb{R} \to \mathbb{R}$..

<u>FINDING $g^{-1}(y)$:</u>

$$g^{-1}(y) = x \Longleftrightarrow g(x) = y \Rightarrow \sqrt[3]{5x-1} = y$$

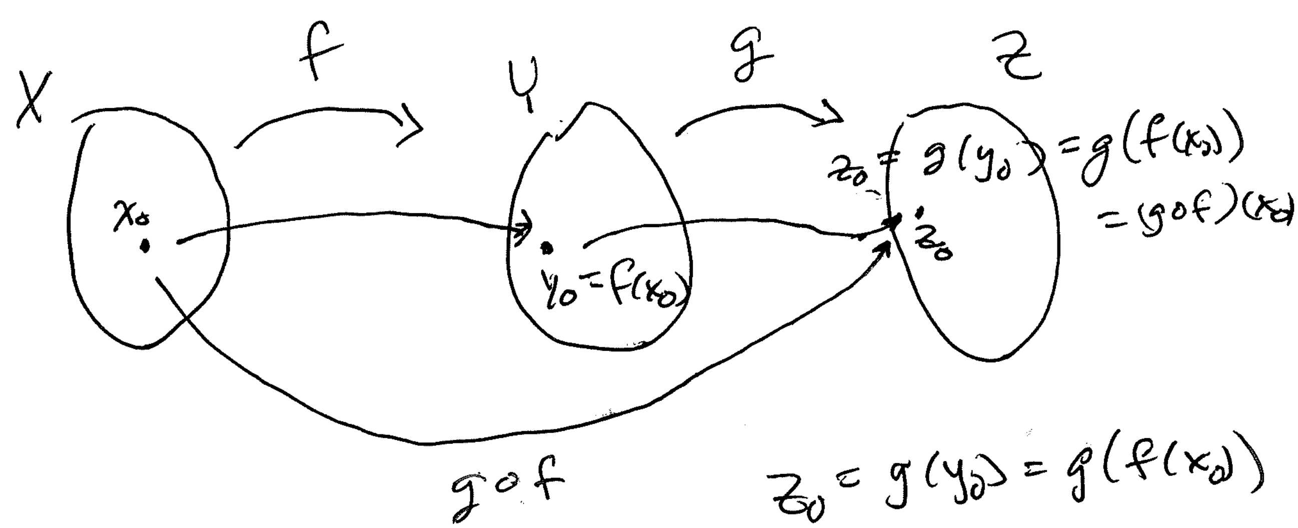
$$y^3 = 5x - 1 \Rightarrow y^3 + 1 = 5x$$

$$x = \frac{1}{5}(y^3+1) = \frac{y^3+1}{5} = g^{-1}(y)$$

Sec 7.3: Composition of Functions

Def'n: Let $f: X \to Y$ and $g: Y \to Z$ be functions such that (RANGE of f) $\subseteq$ (Domain of g) $= Y$, the composition of f and g (the composition of f followed by g) is the function $g \circ f : X \to Z$, defined by the rule:

For each $x \in X$, $g \circ f(x) = g(f(x))$.



$z_0 = g(y_0) = g(f(x_0))$
$= g \circ f(x_0)$

Ex: Define $h: \mathbb{R} \to \mathbb{R}$ and $l: \mathbb{R} \to \mathbb{R}$ as follows:

For all $x \in \mathbb{R}$, $h(x) = x^2 + 5$ and
For all $y \in \mathbb{R}$, $l(y) = \cos(y)$.

$\therefore\ l \circ h(x) = l(h(x)) = l(x^2+5) = \cos(x^2+5) = l \circ h(x)$

$\therefore\ h \circ l(x) = h(l(x)) = h(\cos x) = \cos^2 x + 5 = h \circ l(x)$

$l \circ h \neq h \circ l$.

Theorem 7.3.3 AND Theorem 7.3.4,

Theorem 7.3.3

If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one functions, then $g \circ f$ is one-to-one.

Theorem 7.3.4

If $f: X \to Y$ and $g: Y \to Z$ are both onto functions, then $g \circ f$ is onto.

Def'n: Let X be any set.

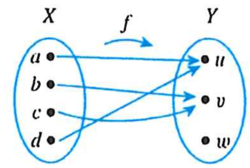
Define $I_X : X \to X$ as follows: For all $t \in X$, $I_X(t) = t$.
(i_X) The Identity function on set X.

Thm 7.3.1: If $f : X \to Y$, $I_Y \circ f = f$
and $f \circ I_X = f$

For a function $f : X \to Y$, a one-to-one correspondence
$f^{-1} : Y \to X$ such that
for all $y \in Y$, $f^{-1}(y) = x \Leftrightarrow f(x) = y$.
When f^{-1} exists, $f^{-1} \circ f = I_X$ and
$f \circ f^{-1} = I_Y$

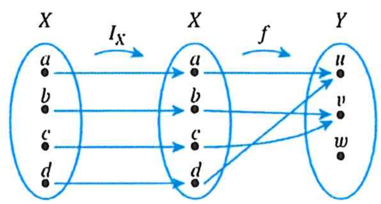
Example 7.3.3 Composition with the Identity Function

Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, and suppose $f: X \to Y$ is given by the arrow diagram shown below.



Find $f \circ I_X$ and $I_Y \circ f$.

Solution The values of $f \circ I_X$ are obtained by tracing through the arrow diagram shown below.



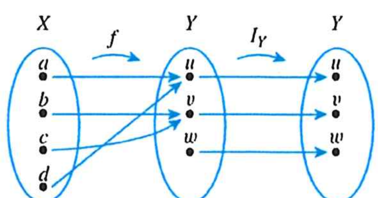
$$(f \circ I_X)(a) = f(I_X(a)) = f(a) = u$$
$$(f \circ I_X)(b) = f(I_X(b)) = f(b) = v$$
$$(f \circ I_X)(c) = f(I_X(c)) = f(c) = v$$
$$(f \circ I_X)(d) = f(I_X(d)) = f(d) = u$$

Note that for all elements x in X,

$$(f \circ I_X)(x) = f(x).$$

By definition of equality of functions, this means that $f \circ I_X = f$.

Similarly, the equality $I_Y \circ f = f$ can be verified by tracing through the arrow diagram below for each x in X and noting that in each case, $(I_Y \circ f)(x) = f(x)$.



More generally, the composition of any function with an identity function equals the function.

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Theorem 7.3.1 Composition with an Identity Function

If f is a function from a set X to a set Y, and I_X is the identity function on X, and I_Y is the identity function on Y, then

$$\text{(a)}\ f \circ I_X = f \quad \text{and} \quad \text{(b)}\ I_Y \circ f = f.$$

Proof:

Part (a): Suppose f is a function from a set X to a set Y and I_X is the identity function on X. Then, for all x in X,

$$(f \circ I_X)(x) = f(I_X(x)) = f(x).$$

Hence, by definition of equality of functions, $f \circ I_X = f$, as was to be shown.

Part (b): This is exercise 13 at the end of this section.

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Theorem 7.3.2 Composition of a Function with Its Inverse

If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$, then

$$\text{(a) } f^{-1} \circ f = I_X \quad \text{and} \quad \text{(b) } f \circ f^{-1} = I_Y.$$

Proof:

Part (a): Suppose $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$. *[To show that $f^{-1} \circ f = I_X$, we must show that for all $x \in X$, $(f^{-1} \circ f)(x) = x$.]* Let x be any element in X. Then

$$(f^{-1} \circ f)(x) = f^{-1}(f(x))$$

by definition of composition of functions. Now the inverse function f^{-1} satisfies the condition

$$f^{-1}(b) = a \quad \Leftrightarrow \quad f(a) = b \quad \text{for all } a \in X \text{ and } b \in Y. \tag{7.3.1}$$

Let

$$x' = f^{-1}(f(x)). \tag{7.3.2}$$

Apply property (7.3.1) with x' playing the role of a and $f(x)$ playing the role of b. Then

$$f(x') = f(x).$$

But since f is one-to-one, this implies that $x' = x$. Substituting x for x' in equation (7.3.2) gives

$$x = f^{-1}(f(x)).$$

Then by definition of composition of functions,

$$(f^{-1} \circ f)(x) = x,$$

as was to be shown.

Part (b): This is exercise 14 at the end of this section.

Composition of One-to-One Functions

The composition of functions interacts in interesting ways with the properties of being one-to-one and onto. What happens, for instance, when two one-to-one functions are composed? Must their composition be one-to-one? For example, let $X = \{a, b, c\}$, $Y = \{w, x, y, z\}$, and $Z = \{1, 2, 3, 4, 5\}$, and define one-to-one functions $f: X \to Y$ and $g: Y \to Z$ as shown in the arrow diagrams of Figure 7.3.1.

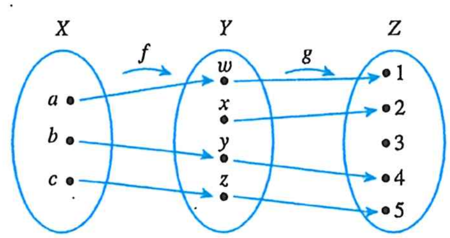


Figure 7.3.1

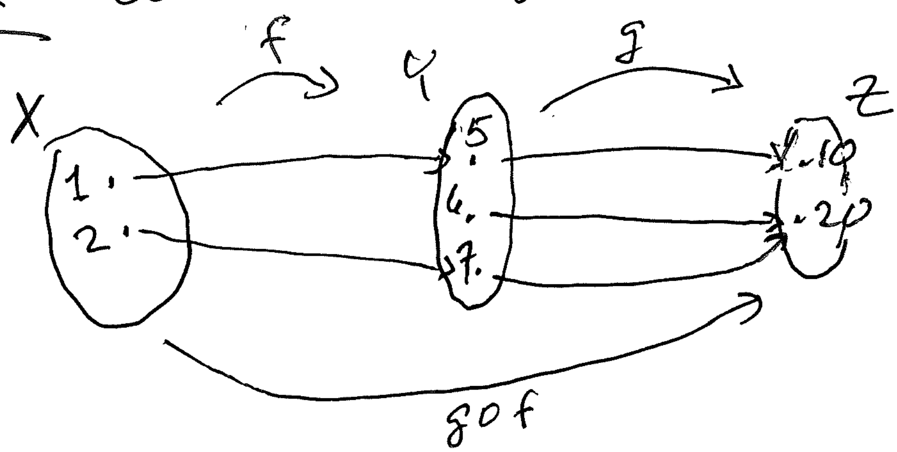
Theorem (N1B).9: With $f: X \to Y$ and $g: Y \to Z$

and so $g \circ f: X \to Z$,

(1) If $g \circ f$ is one-to-one, then f is one-to-one
(but maybe g is not one-to-one)

(2) If $g \circ f$ is onto, then g is onto
(but maybe f is not onto).

Ex: Let $X = \{1, 2\}$, $Y = \{5, 6, 7\}$, $Z = \{10, 20\}$



$g \circ f(1) = g(f(1)) = g(5) = 10$ $\qquad g \circ f(1) = 10$
$g \circ f(2) = g(f(2)) = g(7) = 20$ $\qquad g \circ f(2) = 20$

$g \circ f$ is one-to-one but g is not one-to-one.

$g \circ f$ is onto, but f is not onto.

More on One-to-one Functions and Onto Functions and One-to-one and Onto Functions

Theorem (NIB) 9 (Solutions to Sec 7.3, #18 and #19):

Let $f: X \to Y$ and $g: Y \to Z$ be functions. Then, $g \circ f: X \to Z$.

1) If $g \circ f$ is one-to-one, then f is one-to-one. (f is applied *first.*)

2) If $g \circ f$ is onto, then g is onto. (g is applied *last.*)

Proof: (by contraposition)

1) Suppose that f is not one-to-one. [NTS : $g \circ f$ is not one-to-one.]
$\therefore$ There exist elements $u \in X$ and $v \in X$ are such that
$f(u) = f(v)$ and $u \neq v$. $\therefore g(f(u)) = g(f(v))$ and $u \neq v$.
$\therefore (g \circ f)(u) = (g \circ f)(v)$ and $u \neq v$. $\therefore (g \circ f)$ is not one-to-one.
$\therefore$ If $(g \circ f)$ is one-to-one, then f is one-to-one.

2) Suppose that g is not onto. [NTS: $g \circ f$ is not onto.]
$\therefore$ There exists an element $z_0 \in Z$ such that $g(y) \neq z_0$, for all $y \in Y$.
Suppose x is any element of X. Let $y_0 = f(x) \in Y$.
$\therefore g(y_0) \neq z_0$. $\therefore g(f(x)) \neq z_0$. $\therefore (g \circ f)(x) \neq z_0$.
$\therefore \forall x \in X, (g \circ f)(x) \neq z_0$. $\therefore (g \circ f)$ is not onto.
$\therefore$ If $(g \circ f)$ is onto, then g is onto.
QED

Theorem (NIB) 10: Let $f: X \to Y$ and $g: Y \to X$ be functions.

If $g \circ f = i_X$ and $f \circ g = i_Y$,

then f is a one-to-one correspondence and $g = f^{-1}$.

Proof: Recall: $i_X(x) = x, \forall x \in X$, and $i_Y(y) = y, \forall y \in Y$.

Since i_X is one-to-one and since f is applied first in $g \circ f$,
f is one-to-one by Part 1) of Theorem (NIB) 4.
Since i_Y is onto and since f is applied last in $f \circ g$,
f is onto Part 2) of Theorem (NIB) 4.
$\therefore f$ is a one-to-one correspondence.

The proof that $g = f^{-1}$ is left as an exercise. (#25 of Sec. 7.3) Q E D

An EXAMPLE OF APPLYING THM (NIB) 10 to

Prove a function is a one-to-one correspondence

Ex: Define $f: \mathbb{R} \to \mathbb{R}^+$ as follows:

for all $x \in \mathbb{R}$, $f(x) = 5e^{3x}$.

TO Prove: f is a one-to-one correspondence using Theorem (NIB) 10:

Proof: Let $g: \mathbb{R}^+ \to \mathbb{R}$ be defined by the rule for all $y \in \mathbb{R}^+$,

$g(y) = \frac{1}{3} \ln\left(\frac{y}{5}\right)$.

Workspace
$y = 5e^{3x}$
$\frac{1}{5}y = e^{3x}$
$\ln\left(\frac{y}{5}\right) = 3x$
$x = \frac{1}{3}\ln\left(\frac{y}{5}\right)$

[Showing that $g \circ f = I_{\mathbb{R}}$ and $f \circ g = I_{\mathbb{R}^+}$]

[Showing $g \circ f = I_{\mathbb{R}}$]

Let $x \in \mathbb{R}$ be given. $g \circ f(x) = g(f(x)) = g(5e^{3x})$

$g \circ f(x) = g(5e^{3x}) = \frac{1}{3}\ln\left(\frac{\cancel{5}e^{3x}}{\cancel{5}}\right) = \frac{1}{3}\ln(e^{3x}) = \frac{1}{3} \cdot 3x$

$g \circ f(x) = \frac{1}{3}(3x) = x = I_{\mathbb{R}}(x)$

For all $x \in \mathbb{R}$, $g \circ f(x) = I_{\mathbb{R}}(x)$. So $g \circ f = I_{\mathbb{R}}$.

Showing that $f \circ g = I_{\mathbb{R}^+}$ is left as an exercise.

$\therefore$ $g \circ f = I_{\mathbb{R}}$ and $f \circ g = I_{\mathbb{R}^+}$.

$\therefore$ By THM (NIB) 10, f is a one-to-one correspondence and $g = f^{-1}$.